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CONTROL SYNTHESIS IN A NON-LINEAR DYNAMICAL SYSTEM†

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A non-linear controllable dynamical system of general form, described by Lagrange's equations, is considered. The generalized control forces are subject to geometrical constraints. It is required to construct feedback-implementable control forces that will steer the system in finite time from an arbitrary initial state to a given terminal state. The problem has an explicit solution under fairly general assumptions. The construction utilizes a decomposition of the system into several simpler subsystems, each with one degree of freedom. It is shown, in particular, that if the system is subject to control forces alone, it can be steered in finite time to any given state, however weak these forces. Upper bounds are obtained for the duration of the control process.

1. STATEMENT OF THE PROBLEM

CONSIDER a non-linear dynamical system described by Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}}-\frac{\partial T}{\partial q_i}=Q_i+F_i \tag{1.1}$$

Throughout this paper the dot stands for differentiation with respect to time t, $q=(q_1,\ldots,q_n)$ is the vector of generalized coordinates, T is the kinetic energy of the system, Q_i are uncontrollable generalized forces and F_i are controllable generalized forces. We shall assume that all relevant motions of system (1.1) take place in a domain D in n-space R^n, so that $q\in D$ always. In particular, D may be all of R^n. Throughout, the indices i, j, k take values $1, 2, \ldots, n$.

We will now state our initial assumptions concerning the kinetic energy

$$T=\frac{1}{2}(A(q)q^{\cdot},q^{\cdot})=\frac{1}{2}\sum_{i,j}a_{ij}(q)q_i^{\cdot}q_j^{\cdot} \tag{1.2}$$

of the system and the generalized forces. Here $A(q)$ is a symmetric positive definite $n\times n$ matrix with elements $a_{ij}(q)$ which are continuously differentiable functions of q for $q\in D$. The summation in (1.2) and throughout what follows is performed over values of i, j ranging from 1 to n. It is assumed that for any $q\in D$ all the eigenvalues of $A(q)$ lie in an interval $[m, M]$, where $M>m>0$. Thus, for any n-vector z

$$m(z,z)\leqslant(A(q)z,z)\leqslant M(z,z),\quad 0<m<M,\quad \forall q\in D \tag{1.3}$$

In addition, we will assume that

$$|\partial a_{ij}(q)/\partial q_k|\leqslant C,\quad \forall q\in D,\quad C=\text{const}>0 \tag{1.4}$$

and that the uncontrollable generalized forces Q_i in (1.1) consist of three terms, each subject to different restrictions:

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$$Q_i = P_i + R_i + S_i \tag{1.5}$$

The forces $P_i(q, q^{\bullet}, t)$ are given functions of the generalized coordinates and time.

The terms $R_i(q, q^{\bullet}, t)$ in (1.5) represent dissipative forces. The exact form of $R_i(q, q^{\bullet}, t)$ may be unknown. Our only requirement is that these forces possess the property of dissipativeness, and that they be sufficiently small at low velocities. The former property means that the power of the dissipative forces is non-positive:

$$\sum_i R_i q_i^{\bullet} \leqslant 0 \tag{1.6}$$

for all $q \in D$, all $q^{\bullet}$ and all $t \geqslant t_0$, where t_0 is the starting time. The second property may be stated as follows: there exists a sufficiently small number of $\nu_0 > 0$ such that, if $|q_i^{\bullet}| \leqslant \nu \leqslant \nu_0$ for all i, then

$$|R_i| \leqslant R_i^0(\nu) \tag{1.7}$$

where $R_i^0(\nu)$ are certain continuous monotone increasing functions defined over $\nu \in [0, \nu_0]$, such that $R_i^0(0) = 0$.

The terms $S_i(q, q^{\bullet}, t)$ in (1.5) represent uncertain external perturbations, the only restriction being that they be bounded

$$|S_i| \leqslant S_i^0 \tag{1.8}$$

for all $q \in D$, all $q^{\bullet}$ and $t \geqslant t_0$. Here $S_i^0 > 0$ are specified constants.

Concerning the control forces F_i in (1.1) we will assume that they are large enough to balance the given external forces P_i, after that the control may be chosen in a certain domain. Thus, we assume that F_i can be written as

$$F_i = -P_i(q, q^{\bullet}, t) + G_i \tag{1.9}$$

The vector $G = (G_1, \ldots, G_n)$ may be chosen from some set W, which will generally depend on q, $q^{\bullet}$ and t, i.e.

$$G \in W(q, q^{\bullet}, t) \subset R^n \tag{1.10}$$

We will assume that for all $q \in D$, all $q^{\bullet}$ and all $t \geqslant t_0$ the set W contains a neighbourhood W_0 of the origin:

$$W(q, q^{\bullet}, t) \supset W_0, \quad 0 \in W_0 \tag{1.11}$$

We will assume that W_0 is either a sphere of radius $r > 0$

$$W_0 = \{G : |G| \leqslant r\} \tag{1.12}$$

or a rectangular parallelepiped, corresponding to independent constraints on G

$$W_0 = \{G : |G_i| \leqslant G_i^0\} \tag{1.13}$$

In the case of constraints (1.13) we define

$$r = \min_i G_i^0 \tag{1.14}$$

Substituting (1.5) and (1.9) into system (1.1), we get

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\bullet}} - \frac{\partial T}{\partial q_i} = R_i + S_i + G_i \tag{1.15}$$

Suppose we are given initial conditions

$$q(t_0) = q^0, \quad q^{\bullet}(t_0) = (q^{\bullet})^0 \tag{1.16}$$

and terminal conditions corresponding to the state of rest

$$q(t_*) = q^*, \quad q^{\bullet}(t_*) = 0 \tag{1.17}$$

where $q^0 \in D$, $q^* \in D$, $t_* > t_0$. The control problem may be formulated as follows.

Problem 1. It is required to find a feedback-implementable control $G(q, q^\bullet)$ that satisfies the condition

$$G \in W_0 \tag{1.18}$$

and steers system (1.15) from any initial state (1.16) to a given terminal state (1.17) in a finite (but not fixed) time. The set W_0 is given as (1.12) or (1.13) and in either case, by (1.14), contains the sphere $|G| \leqslant r$. The kinetic energy of system (1.15) is defined by (1.2) and satisfies conditions (1.3) and (1.4), while the forces R_i and S_i in (1.15) satisfy the constraints (1.6)–(1.8).

Note that if the control G satisfies constraint (1.18), it follows from (1.11) that it also satisfies the initial constraint (1.10).

We will first construct a solution of Problem 1 on the assumption that system (1.15) involves no dissipative forces or perturbations, that is, $R_i = S_i = 0$. The general case will be considered later.

2. CONTROL WHEN THERE ARE NO EXTERNAL FORCES

If $R_i = S_i = 0$, system (1.15) becomes

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^\bullet} - \frac{\partial T}{\partial q_i} = G_i \tag{2.1}$$

We will let $\varepsilon > 0$ be some given positive number and let Ω_1 denote the set of all points of the $2n$-dimensional phase space $(q, q^\bullet)$ at which $q \in D$ and $|q_i^\bullet| > \varepsilon$ for at least one i. Let Ω_2 denote the set of all points $(q, q^\bullet)$ at which $q \in D$ and $|q_i^\bullet| \leqslant \varepsilon$ for all i. Thus,

$$\begin{gathered}\Omega_1 = \{(q, q^\bullet):\ q \in D;\quad \exists i,\ |q_i^\bullet| > \varepsilon\}\\ \Omega_2 = \{(q, q^\bullet):\ q \in D;\quad \forall i,\ |q_i^\bullet| \leqslant \varepsilon\}\end{gathered} \tag{2.2}$$

We will construct the control $G(q, q^\bullet)$ separately for each of the domains Ω_1 and Ω_2, and also specify the number ε. By the theorem on the variation of kinetic energy, applied to system (2.1), we have

$$dT/dt = \sum_i G_i q_i^\bullet = (G, q^\bullet) \tag{2.3}$$

We will choose a control Ω in Ω_1 so as to satisfy the constraints (1.18) and so that the derivative (2.3) is negative. To that end we define

$$G = -rq^\bullet |q^\bullet|^{-1}, \qquad G_i = -G_i^0 \operatorname{sign} q_i^\bullet \tag{2.4}$$

for cases (1.12) and (1.13), respectively. Substituting (2.4) into (2.3), we obtain, respectively,

$$dT/dt = -r|q^\bullet|, \quad dT/dt = -\sum_i G_i^0 |q_i^\bullet| \tag{2.5}$$

In view of the notation (1.14), we see that in both cases (1.12), (1.13),

$$dT/dt = 2T^{1/2} dT^{1/2}/dt \leqslant -r|q^\bullet| \tag{2.6}$$

The upper bound (1.3) for the kinetic energy gives

$$|q^\bullet| \geqslant (2T/M)^{1/2} \tag{2.7}$$

Substituting (2.7) into the right-hand side of inequality (2.6) and noting that $T > 0$ in Ω_1 [see (2.2)], we obtain

$$dT^{1/2}/dt \leqslant -r(2M)^{-1/2} \tag{2.8}$$

Integrating inequality (2.8), we have

$$T^{1/2}-T_0^{1/2}\leqslant -r(2M)^{-1/2}(t-t_0) \tag{2.9}$$

where T_0 is the kinetic energy at the starting time t_0. It follows from (2.9) that in a finite time the kinetic energy will become as small as desired. Consequently, at some time t_1 the system will reach the border between Ω_1 and Ω_2.

We shall need bounds for the time t_1 and generalized coordinates $q(t_1)$. By (1.3) and (2.2), if T_1 is the kinetic energy at time t_1, then

$$T_1\geqslant m(q^{\cdot}, q^{\cdot})/2\geqslant m\varepsilon^2/2 \tag{2.10}$$

Inequalities (2.9) and (2.10) yield the required bound for t_1:

$$t_1-t_0\leqslant\tau_1, \quad \tau_1=(2M)^{1/2}r^{-1}[T_0^{1/2}-(m/2)^{1/2}\varepsilon] \tag{2.11}$$

To estimate $q(t_1)$, we write the obvious inequalities

$$|q_i(t_1)-q_i^0|\leqslant\int_{t_0}^{t_1}|q_i^{\cdot}|\,dt\leqslant\int_{t_0}^{t_1}|q^{\cdot}|\,dt \tag{2.12}$$

We will use the following inequalities, which follow from (1.3) and (2.9):

$$|q^{\cdot}|\leqslant(2T/m)^{1/2}\leqslant(2/m)^{1/2}[T_0^{1/2}-r(2M)^{-1/2}(t-t_0)] \tag{2.13}$$

Substituting (2.13) into (2.12) and integrating, we obtain

$$|q_i(t_1)-q_i^0|\leqslant\varphi(t_1-t_0),$$
$$\varphi(\tau)=(2T_0/m)^{1/2}\tau-r(Mm)^{-1/2}\tau^2/2 \tag{2.14}$$

A direct check will show that $\varphi(\tau)$ is a strictly increasing function in the interval $[0, \tau_1]$, where τ_1 is defined in (2.11). Since $t_1-t_0\leqslant\tau_1$ [see (2.11)], it follows that $\varphi(t_1-t_0)\leqslant\varphi(\tau_1)$, and therefore, using (2.11), we deduce from (2.14) that

$$|q_i(t_1)-q_i^0|\leqslant\varphi(\tau_1)=(M/m)^{1/2}r^{-1}(T_0-m\varepsilon^2/2) \tag{2.15}$$

Thus, at a time t_1 the system is on the boundary of Ω_1 and Ω_2. We construct the control in Ω_2 so that the system, having once entered Ω_2, will never leave it again but will reach the terminal state (1.17) in a finite time.

We will write Lagrange's equations (2.1) in expanded form, substituting T from (1.2):

$$\sum_j a_{ij}q_j^{\cdot\cdot}+\sum_{j,k}\gamma_{ijk}q_j^{\cdot}q_k^{\cdot}=G_i \tag{2.16}$$

Here

$$\gamma_{ijk}=\frac{\partial a_{ij}}{\partial q_k}-\frac{1}{2}\frac{\partial a_{jk}}{\partial q_i} \tag{2.17}$$

where γ_{ijk} may be regarded as the components of n-vectors

$$\Gamma_{jk}=(\gamma_{1jk}, \ldots, \gamma_{njk}) \tag{2.18}$$

We rewrite Eq. (2.16) in vector notation and solve it for $q^{\cdot\cdot}$. This gives

$$q^{\cdot\cdot}=U+V \tag{2.19}$$

where

$$U=A^{-1}G, \quad V=-\sum_{j,k}A^{-1}\Gamma_{jk}q_j^{\cdot}q_k^{\cdot} \tag{2.20}$$

It follows from condition (1.3) that the eigenvalues of the inverse A^{-1} lie in the interval $[M^{-1}, m^{-1}]$. Consequently, for any n-vector z,

$$|Az|\leqslant M|z|, \quad |A^{-1}z|\leqslant m^{-1}|z| \tag{2.21}$$

We subject the components U_i of the vector U to the constraints

$$|U_i|\leqslant U^0, \quad U^0=rM^{-1}n^{-1/2} \tag{2.22}$$

The truth of these inequalities implies that of the inequality $|U|\leqslant rM^{-1}$, which, in turn, by (2.20), (2.21), implies $|G|=|AU|\leqslant M|U|\leqslant r$. Consequently, G satisfies (1.18) whether W_0 is taken to be (1.12) or (1.13). Thus the constraint (2.22) implies the truth of condition (1.18).

To estimate the vector V in (2.20), we use the second inequality of (2.21)

$$|V|\leqslant m^{-1}\sum_{j,k}|\Gamma_{jk}||q_j^{\cdot}||q_k^{\cdot}| \tag{2.23}$$

Inequalities (1.4) imply estimates for the quantities γ_{ijk} introduced in (2.17): $|\gamma_{ijk}|\leqslant 3/2C$. Hence, using (2.18), we have

$$|\Gamma_{jk}|=\Big(\sum_i \gamma_{ijk}^2\Big)^{1/2}\leqslant \frac{3}{2}Cn^{1/2}$$

We substitute these bounds for Γ_{jk}, and also the inequalities $|q_i^{\cdot}|\leqslant\varepsilon$—which are true in Ω_2 by virtue of (2.2)—into (2.23). This gives $|V|\leqslant 3/2Cn^{5/2}m^{-1}\varepsilon^2$. Consequently, we have the following bounds for the components V_i of V

$$|V_i|\leqslant V^0, \quad V^0={}^3/_2Cn^{5/2}m^{-1}\varepsilon^2 \tag{2.24}$$

Equations (2.19) and the constraints (2.22) and (2.24) may be rewritten as

$$q_i^{\cdot\cdot}=U_i+V_i, \quad |U_i|\leqslant U^0, \quad |V_i|\leqslant V^0 \tag{2.25}$$

where U^0 and V^0 are as defined in (2.22) and (2.24).

Assuming that

$$\rho=V^0/U^0<1 \tag{2.26}$$

we will construct a control U_i separately for each degree of freedom of system (2.25).

To do this we will admit that V_i may be arbitrary functions satisfying the constraints (2.25). We will use the minimax (guaranteed) approach, which is characteristic of the theory of differential games [1].

Considering the ith equation of (2.25), we define

$$q_i-q_i^*=x, \quad q_i^{\cdot}=x^{\cdot}=y, \quad U_i=u, \quad V_i=v \tag{2.27}$$

and rewrite (2.25) and (2.26) as

$$x^{\cdot}=y, \quad y^{\cdot}=u+v, \quad |u|\leqslant U^0, \quad |v|\leqslant\rho U^0, \quad 0<\rho<1 \tag{2.28}$$

At time t_1, by assumption, the system is at the boundary of the domains Ω_1 and Ω_2 [see (2.2)]. Taking (2.27) into account, we have the following initial conditions for system (2.28):

$$x(t_1)=x^1=q_i(t_1)-q_i^*, \quad y(t_1)=y^1=q_i^{\cdot}(t_1), \quad |y^1|\leqslant\varepsilon \tag{2.29}$$

The terminal conditions (1.17) become

$$x(t_*)=0, \quad y(t_*)=0 \tag{2.30}$$

To ensure that the system, having reached Ω_2 at time t_1, will not leave the domain again, we require that

$$|y(t)|\leqslant\varepsilon, \quad t>t_1 \tag{2.31}$$

Thus, we have the following decomposition of Problem 1 in Ω_2: instead of the problem for the initial system with n degrees of freedom, we obtain n analogous problems for systems with one degree of freedom each. To solve Problem 1 in Ω_2, therefore, we need only solve the following problem.

Problem 2. Find a control $u(x, y)$ for system (2.28) that satisfies the constraints (2.28) and (2.31) and will steer the system from the initial state (2.29) to a terminal state (2.30) in a finite time for any admissible v satisfying (2.28).

Problem 2 will be solved by a certain modification of a method used to solve a simple differential game for system (2.28). The players in the game select controls u and v subject to constraints (2.28); controls u endeavour to decrease and controls v to increase the time t_* at which the system reaches the origin [see (2.30)]. It is known (see [1]) that the optimal control u in this game is the same as the time-optimal control for the system

$$x^{\cdot}=y, \quad y^{\cdot}=(1-\rho)u, \quad |u|\leqslant U^0 \tag{2.32}$$

This system is derived from (2.28) by putting $v = -\rho u$, which corresponds to an optimal (worst for u) control for the second player, who selects v. The synthesis of a time-optimal control for system (2.32) with terminal condition (2.30) is determined by the relations

$$\begin{aligned} &u(x, y)=U^0 \operatorname{sign}[\psi_0(x)-y] \quad \text{if} \quad y\neq\psi_0(x) \\ &u(x, y)=U^0 \operatorname{sign} x=-U^0 \operatorname{sign} y \quad \text{if} \quad y=\psi_0(x) \end{aligned} \tag{2.33}$$

where

$$\psi_0(x)=-[2U^0(1-\rho)|x|]^{1/2} \operatorname{sign} x \tag{2.34}$$

The switching curve $y = \psi_0(x)$ for the control (2.33) is the union of two branches of parabolas which are symmetrical about the origin. These branches are the optimal trajectories that reach the origin.

Unlike this solution of problem (2.33), (2.34), our solution of Problem 2 must take the phase constraint (2.31) into consideration. During the construction, however, we can drop the optimality condition.

Choose some number $\delta\in(0, \varepsilon)$ and define a function $\psi(x)$ by the equations

$$\psi(x) = \begin{cases} \psi_0(x), & |x|\leqslant x^* \\ -\delta \operatorname{sign} x, & |x|>x^* \end{cases} \tag{2.35}$$

The number x^* in (2.35) is determined from the condition that $\psi(x)$ be a continuous function, i.e. $\psi_0(x^*) = -\delta$. By (2.34), we get

$$x^*=\delta^2[2U^0(1-\rho)]^{-1} \tag{2.36}$$

The required solution $u(x, y)$ of Problem 2 may be written in the form (2.33), with $\psi_0(x)$ replaced by $\psi(x)$ as defined in (2.35). We have

$$u(x, y) = \begin{cases} U^0 \operatorname{sign}[\psi(x)-y], & y\neq\psi(x) \\ U^0 \operatorname{sign} x = -U^0 \operatorname{sign} y, & y=\psi(x) \end{cases} \tag{2.37}$$

The switching curve $y = \psi(x)$ for the control (2.37), (2.35) is symmetrical about the origin and is the union of two arcs of parabolas and two rays (the solid curve in Fig. 1). Since $\delta<\varepsilon$, this curve lies within the strip $|y|\leqslant\varepsilon$ and divides it into two symmetrical parts: the domain X^+ where $y<\psi(x)$ and $u = U^0$, and the domain X^- where $y>\psi(x)$, $u = U^0$ [see (2.37)].

We shall prove that the control (2.37), (2.35) solves Problem 2.

The initial conditions (2.29) hold at time t_1.

According to Eqs (2.28) and the control law (2.37), we have

$$y^{\cdot}\geqslant U^0(1-\rho), \quad (x, y)\in X^+; \quad y^{\cdot}\leqslant -U^0(1-\rho), \quad (x, y)\in X^- \tag{2.38}$$

The width of the domains X^+, X^- in the y direction is at most $\varepsilon+\delta$ (see Fig. 1), while the velocity of motion in that direction is finite and directed toward the switching curve [see (2.38)].

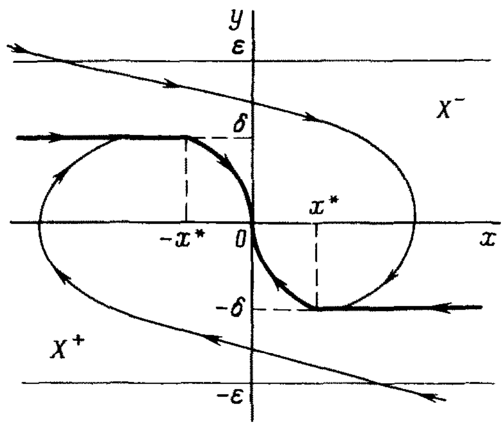

FIG. 1.

Consequently, the phase point will never leave the strip $|y|\leqslant\varepsilon$ but after a finite time, at a certain time $t_2\geqslant t_1$, will be incident on the switching curve $y=\psi(x)$.

Suppose that at time t_2 the phase point has hit the straight part $y=\pm\delta$ of the switching curve $y=\psi(x)$. After that the point will move along the straight part of the curve in a sliding regime. This follows from the fact that the phase velocities on both sides of this part of the curve are finite and directed toward the switching curve. The motion will take place along these parts of the curve at an appropriate constant velocity $y=x^0=\pm\delta$ in the direction of decreasing $|x|$. Consequently, in a finite time, at some time $t_3\geqslant t_2$, the phase point will reach one of the points $(\pm x^*,\ \mp\delta)$ at the junction of the straight and curved parts of the switching curve. The curved (parabolic) parts are the phase trajectories of system (2.28) if u is selected in accordance with (2.37) and $v=-\rho u$. If $v\neq-\rho u$, the motion induced by control (2.37) will nevertheless take place along these parts of the parabolas, but in a sliding regime. In a finite time, therefore, at time t_*, the phase will reach the origin.

The thin curves in the figure represent some possible phase trajectories. The arrows indicate the direction of increasing time t.

The entire motion, from time t_1 to time t_*, falls into three stages: motion in the domain X^+ or X^-; motion along the straight lines $y=\pm\delta$ and motion along the parabolas. Some of these stages may be missing. For example, at the initial time t_1 the phase point may either lie on the switching curve or proceed directly from X^+ or X^- to the parabolic part of the curve. In all cases, however, the duration t_*-t_1 of the motion is finite.

To estimate this total time, let us assume that all three stages actually occur—this will lead us to an upper bound. The length t_2-t_1 of the first stage (motion in X^+ or X^-) is estimated by dividing the maximum width $\varepsilon+\delta$ of the domains along the y axis by the velocity $\dot{y}$ of minimum absolute value as in (2.38). This gives

$$t_2-t_1\leqslant(\varepsilon+\delta)[U^0(1-\rho)]^{-1} \tag{2.39}$$

To estimate the coordinate $x(t_2)$ we will use the constraint (2.31) and the initial condition (2.29):

$$|x(t_2)-x^1|\leqslant\int_{t_1}^{t_2}|y|\,dt\leqslant\varepsilon(t_2-t_1)$$

Hence, using (2.39), we get

$$|x(t_2)|\leqslant|x^1|+\varepsilon(\varepsilon+\delta)[U^0(1-\rho)]^{-1} \tag{2.40}$$

The length t_3-t_2 of the second stage (motion along straight lines $y=\pm\delta$) is estimated by dividing the distance along the x axis by the velocity, which is δ in absolute value

$$t_3-t_2\leqslant[|x(t_2)|-x^*]\delta^{-1}$$

Substituting (2.36) and (2.40) into this inequality, we obtain

$$t_3-t_2\leqslant|x^1|\delta^{-1}+\varepsilon(\varepsilon+\delta)[U^0(1-\rho)\delta]^{-1}-\delta[2U^0(1-\rho)]^{-1} \tag{2.41}$$

The length t_*-t_3 of the third, last stage (parabolic motion) may be estimated by dividing the velocity δ of maximum absolute value at the beginning of the stage by the acceleration of minimum absolute value, defined by (2.38). This gives

$$t_*-t_3\leqslant\delta[U^0(1-\rho)]^{-1} \tag{2.42}$$

Adding (2.39), (2.41) and (2.42), we obtain an upper bound for the total duration of motion in Problem 2:

$$t_*-t_1\leqslant|x^1|\delta^{-1}+(2\varepsilon^2+4\varepsilon\delta+3\delta^2)\delta^{-1}[2U^0(1-\rho)]^{-1} \tag{2.43}$$

The result may be summarized in the form of a theorem.

Theorem 1. The control $u(x,y)$ determined by Eqs (2.37) and (2.35), in which the function ψ_0 and number x^* are defined by (2.34), (2.36) and δ by any number in the interval $(0,\varepsilon)$, is a solution of Problem 2, i.e. it satisfies the constraints (2.28), (2.31) and steers system (2.28) from the initial state (2.29) to the terminal state (2.30) in a finite time t_*-t_1 which is bounded as in (2.43).

We now turn to the solution of the original Problem 1 in the case $R_i=S_i=0$. The required control $G(q,q^{\cdot})$ in Ω_1 is defined by (2.4); the control in Ω_2 may be obtained from the solution $u(x,y)$ of Problem 2. To that end it is sufficient to use the relations $G=AU$ of (2.20) and the notation (2.27). The result is

$$G(q,q^{\cdot})=A(q)U(q,q^{\cdot}),\quad U_i(q_i,q_i^{\cdot})=u(q_i-q_i^*,q_i^{\cdot}) \tag{2.44}$$

We recall that the solution $u(x,y)$ of Problem 1 was obtained on the assumption that $\rho<1$ [see (2.26)]; with the notation (2.22) and (2.24), this leads to the following restriction on ε:

$$\varepsilon<\varepsilon_0=(2mr)^{1/2}(3MCn^3)^{-1/2} \tag{2.45}$$

To estimate the total duration t_*-t_1 of the motion we must add the times of motion in the domains Ω_1 and Ω_2. When evaluating t_*-t_1 we take into consideration that $|x^1|$ in (2.43) should be replaced by the maximum (over i) difference $|q_i(t_1)-q_i^*|$ [see (2.29)], since the system will reach the terminal state when all coordinates take their terminal values. Using the estimate (2.15), we obtain

$$|x^1|=\max_i|q_i(t_1)-q_i^*|\leqslant\max_i(|q_i(t_1)-q_i^0|+|q_i^0-q_i^*|)\leqslant$$
$$\leqslant\max_i|q_i^0-q_i^*|+(M/m)^{1/2}r^{-1}(T_0-m\varepsilon^2/2)$$

This expression is substituted into (2.43), which we then add to inequality (2.11):

$$\begin{aligned}t_*-t_0\leqslant\delta^{-1}\max_i|q_i^0-q_i^*|+(2M)^{1/2}r^{-1}[T_0^{1/2}-(m/2)^{1/2}\varepsilon]+\\+(M/m)^{1/2}r^{-1}\delta^{-1}(T_0-m\varepsilon^2/2)+\\+(2\varepsilon^2+4\varepsilon\delta+3\delta^2)\delta^{-1}[2U^0(1-\rho)]^{-1}\end{aligned} \tag{2.46}$$

The parameters U^0 and ρ are defined by (2.22), (2.24) and (2.26), with $\rho>1$ by condition (2.45).

The result may be stated as the following theorem.

Theorem 2. Problem 1 for system (2.1), i.e. when $R_i=S_i=0$, is always solvable. For any $\varepsilon\in(0,\varepsilon_0)$, with ε_0 given by (2.45), the control $G(q,q^{\cdot})$ defined by (2.4) in Ω_1 (for cases (1.12) and (1.13), respectively) and by (2.44) in Ω_2 solves the problem, i.e. it steers system (2.1) from any initial state (1.16) to a given terminal state (1.17) in a finite time t_*-t_0 which satisfies inequality (2.46). Under these conditions the function $u(x,y)$ in (2.44) is defined by (2.37), (2.35) and (2.34), in which the parameters U^0, ρ and x^* are given by formulas (2.22), (2.26), (2.24) and (2.36) and δ is any number in the interval $(0,\varepsilon)$.

We observe that, in order to reduce the duration of the motion, δ should be chosen as close as possible to ε. If $\delta=\varepsilon$, however, one can no longer guarantee that the system, after reaching the boundary of Ω_1 and Ω_2, will remain in Ω_2. For that reason δ should be chosen in the interval $(0,\varepsilon)$. Our solutions to Problems 1 and 2 are naturally not unique. In particular, there are other possible ways to synthesize controls in the one-dimensional system (2.28) obtained by the above decomposition.

3. CONTROL OF THE GENERAL CASE

We now proceed to solve Problem 1 for system (1.15) in the general case. The approach is largely the same as in Sec. 2.

Letting $\varepsilon > 0$ be given, we again introduce the domains Ω_1 and Ω_2 defined by (2.2). By the dissipative property (1.6) of the forces R_i, the theorem on the variation of the kinetic energy of system (1.15) yields a relation similar to (2.3)

$$dT/dt \leqslant \sum_i (G_i + S_i) q_i^{\cdot} \tag{3.1}$$

The control G in Ω_1 will be chosen so as to minimize the scalar product $(G, q^{\cdot})$ subject to the constraint (1.18). Whether W_0 is defined by (1.12) or (1.13), we again obtain the appropriate expression of (2.4). We now substitute these expressions into inequality (3.1) and use the constraints (1.8), as well as the Cauchy inequality. If W_0 is a sphere (1.12), we obtain

$$dT/dt \leqslant -r|q^{\cdot}| + \sum_i S_i^0 |q_i^{\cdot}| \leqslant -r_1 |q^{\cdot}| \tag{3.2}$$

where

$$r_1 = r - \left[\sum_i (S_i^0)^2\right]^{1/2} > 0 \tag{3.3}$$

If W_0 is a rectangular parallelepiped (1.13), we obtain

$$dT/dt \leqslant -\sum_i (G_i^0 - S_i^0) |q_i^{\cdot}| \leqslant -r_2 |q_i^{\cdot}| \tag{3.4}$$

where

$$r_2 = \left[\sum_i (G_i^0 - S_i^0)^2\right]^{1/2} \tag{3.5}$$

and it is assumed that

$$G_i^0 > S_i^0, \quad i = 1, \ldots, n \tag{3.6}$$

Thus, if inequalities (3.3) hold for the sphere (1.12), or inequalities (3.6) for the parallelepiped (1.13), then inequalities (3.2) and (3.4) lead directly to inequality (2.6) with the constant $r>0$ replaced by $r_\alpha > 0$. Here and below the indices $\alpha = 1, 2$ correspond to cases (1.12) and (1.13), respectively. Hence all the formulas of Sec. 2 relating to Ω_1 remain valid, with the above reservation.

We will now consider Ω_2. We impose the condition

$$\varepsilon \leqslant v_0 \tag{3.7}$$

under which the estimates (1.7) will hold in the domain. Lagrange's equations (1.15) may again be converted to the form (2.19), by solving for the derivatives

$$q^{\cdot\cdot} = U + V_* \tag{3.8}$$

with the same relation (2.20) holding for U as before. The vector V_* in (3.8) is given by

$$V_* = V + A^{-1}(R + S) \tag{3.9}$$

The vector V is defined by (2.20); R and S are the vectors with components R_i and S_i, respectively.

Using inequalities (2.21) for A^{-1}, (1.7) for R_i and (1.8) for S_i, we obtain the estimate

$$|A^{-1}(R+S)| \leqslant m^{-1}[R^0(\varepsilon) + S^0] \tag{3.10}$$

$$R^0(\varepsilon) = \left\{\sum_i [R_i^0(\varepsilon)]^2\right\}^{1/2}, \quad S^0 = \left[\sum_i (S_i^0)^2\right]^{1/2}$$

In accordance with the assumptions made in Sec. 1 about the functions R_i^0 [see (1.7)], $R^0(\varepsilon)$ is a continuous monotone increasing function of ε, with $R^0(0) = 0$.

Inequalities (2.24) and (3.10) imply the following bound for the vector V_* in (3.9):

$$|V_*| \leqslant V_*^0 = V^0 + m^{-1}[R^0(\varepsilon) + S^0] = \\ = m^{-1}[{}^3/{}_2Cn^{3/2}\varepsilon^2 + R^0(\varepsilon) + S^0] \tag{3.11}$$

We impose the following analogue of condition (2.26):

$$\rho^* = V_*^0/U^0 < 1 \tag{3.12}$$

The procedure for constructing the control in Ω_2 and all the subsequent estimates in that domain remain the same as in Sec. 2. The only changes are to replace ρ by ρ^* and r by r_α in the estimates (2.46) for the time. In formula (2.22) for U^0 the parameter r must be retained without change: here it is defined by (1.12) and (1.14) for cases (1.12) and (1.13), respectively. In addition, the restrictions on the choice of ε are changed: instead of (2.45) we now have two conditions: (3.7) and (3.12). In developed form, using (2.22) and (3.11), we obtain

$$\varepsilon \leqslant \nu_0, \quad {}^3/{}_2Cn^{3/2}\varepsilon^2 + R^0(\varepsilon) + S^0 \leqslant mM^{-1}rn^{-1/2} \tag{3.13}$$

Thus, our procedure for control synthesis will produce a solution of Problem 1 provided the following conditions are satisfied: inequalities (3.3) or (3.6) in cases $\alpha = 1, 2$, respectively, and both inequalities (3.13) for ε. A number ε satisfying (3.13) will always exist if there are no perturbations ($S^0 = 0$) or if the perturbations are sufficiently small ($S^0 \leqslant mM^{-1}rn^{-1/2}$). This follows from the continuity of $R^0(\varepsilon)$: $R^0 \to 0$ as $\varepsilon \to 0$. We note that in the case of dissipative forces proportional to the velocities the functions R_i^0 in (1.7) and R^0 in (3.10) are linear in ε.

We summarize the results.

Theorem 3. Let α be 1 or 2, depending on whether W_0 is a sphere (1.12) or a parallelepiped (1.3), respectively. Assume that conditions (3.3), (3.6) are satisfied for $\alpha = 1, 2$, respectively, and that there exists $\varepsilon > 0$ satisfying both conditions (3.13). Then the control $G(q, q^{\cdot})$ defined by (2.4) in Ω_1 (for $\alpha = 1, 2$, respectively) and by (2.44) in Ω_2 solves Problem 1 for system (1.15), i.e. it steers the system from any initial state (1.16) to a given terminal state (1.17). Under these conditions the function $u(x, y)$ in (2.44) is defined by (2.37), (2.35) and (2.34) in which the parameters U^0, x^* are given by (2.22) and (2.36). The parameter ρ in formulas (2.34) and (2.36) should be replaced by ρ^* as in (3.12) and (3.11); under these conditions we have $\rho^* < 1$. The number δ may be chosen anywhere in the interval $(0, \varepsilon)$. The duration $t_* - t_0$ of the motion is finite and satisfies inequality (2.46) with r replaced by r_α [see (3.3), (3.5)] and ρ by ρ^*.

It should be pointed out that similarly formulated control problems have been considered in [2, 3] with more stringent conditions on the magnitude of the control forces. Thus, if there are no external forces ($R_i = S_i = 0$), Sec. 2 of this paper yields a solution of the control problem for control forces as small as desired, whereas in [2, 3] we required that the control forces could take fairly large values. On the other hand, the duration of the control process as obtained in [2, 3] was generally smaller. As shown in [3], the results may be applied to control problems for manipulative robots.

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